Frequency compensated LC networks for oscillators with the wide tuning range.
Theoretical discussion section.
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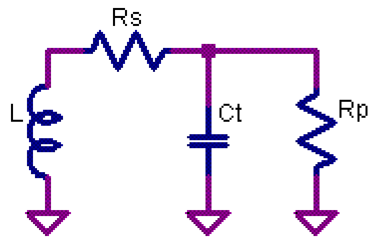
1. Properties of passive, capacitively tuned LC resonator circuits with the Rs-dominant, Rp-dominant and mixed loss models under varying resonance frequency.

Let us first calculate Q-factor $Q(\omega)$ and bandwidth $\Delta\omega(\omega)$ parameter variation with the frequency ω for the linear (resistive) loss model presented by Rs and Rp. Consider the circuit diagram below:



LC circuit resonance frequency *ω* is determined by:

$$\omega=\frac{1}{\sqrt{(L*C)}} \quad (1)$$

In a capacitively tuned oscillator *L* stays constant while both C and ω change. This allows us to express C in terms of ω:

$$C=\frac{1}{L*\omega^2} \quad (2)$$

In a resonant circuit the energy is recirculating between the inductor and the capacitor. For the moments when the tank energy is stored completely in the capacitor or completely in the inductor we can write:

$$E.stored=E.c=E.l$$

The capacitor energy is:

$$E.c=\frac{CV^2}{2}$$ (3), where V is the peak value of the tank voltage in the LC circuit.

The inductor energy is:

$$E.l=\frac{LI^2}{2}$$, where I is the peak value of the tank current in the LC circuit.

These two expressions allow us to determine the peak voltage and current:

$$V^2=\frac{2*E.stored}{C} \quad , \quad I^2=\frac{2*E.stored}{L} \quad (4)$$

The total RMS power dissipation in Rs and Rp can be expressed in terms of the peak V, I as:

$$P.loss=\frac{1}{2}*\frac{V^2}{Rp}+\frac{1}{2}*I^2*Rs$$, and using (4) can be re-written as:

$$P.loss=E.stored*\left(\frac{1}{C*Rp}+\frac{Rs}{L}\right) \quad (5)$$

The definition of Q is:

$$Q=\frac{E.stored}{E.lost.per.cycle} \quad \text{,or} \quad Q=\frac{\omega * E.stored}{P.loss} \quad (6)$$

Therefore using P.loss from (5) we can write:

$$Q(\omega)=\frac{\omega}{\frac{1}{C * Rp}+\frac{Rs}{L}} \quad (7)$$

Note that if we substitute ω from (1) into (7) and then consider Rp and Rs cases separately we arrive to the well known formulas for Q in parallel and series RLC circuits:

$$Q=\frac{C * Rp}{\sqrt{L * C}}=Rp * \sqrt{\frac{C}{L}} \quad , \quad Q=\frac{L}{Rs * \sqrt{L * C}}=\frac{1}{Rs} * \sqrt{\frac{L}{C}}$$

But in this study we are interested in Q-factor dependence on ω in the capacitively tuned circuits, therefore we will instead substitute C from (2) into (7) and write:

$$Q(\omega)=\frac{\omega}{\frac{L * \omega^2}{Rp}+\frac{Rs}{L}} \quad (8)$$

Next we determine how the resonant circuit bandwidth $\Delta\omega(\omega)$ changes with *ω*:

$Q=\frac{\omega}{\Delta\omega}$, hence $\Delta\omega=\frac{\omega}{Q}$, and substituting Q(ω) from (8) we obtain:

$$\Delta\omega(\omega)=\frac{L}{Rp} * \omega^2+\frac{Rs}{L} \quad (9)$$

We now consider two special cases by assigning either Rs=0 or Rp=0 and simplifying (8) and (9):

For the <u>Rp-dominant loss model</u> we get:

$$Q(\omega)=\frac{Rp}{L} * \frac{1}{\omega}$$

$$\Delta\omega(\omega)=\frac{L}{Rp} * \omega^2$$

From these formulas we can make two observations for this loss model:

1. Q(ω) is inversely proportional to ω (decreases with frequency).
2. The bandwidth of a resonant circuit loaded with a parallel resistor Rp increases quadratically with ω.

This has important practical implication - if a fixed bandwidth is required from a capacitively tuned LC circuit over the extended tuning range, and the intrinsic Q of the circuit is excessive, it should never be dampened with parallel resistor. There is a better way described below.

For the <u>Rs-dominant loss model</u> we get:

$$Q(\omega)=\frac{L}{Rs} * \omega$$

$$\Delta\omega=\frac{Rs}{L}$$

Again we can make two important observations for this loss model:

1. Q(ω) is proportional to ω (increases with frequency).
2. The bandwidth of a resonant circuit loaded with a series resistor Rs does not depend on frequency!

This again has an important practical implication - if a fixed finite bandwidth is required from a capacitively tuned LC circuit over the extended tuning range, and the intrinsic Q of the circuit is excessive, then the circuit should be damped with a series resistor.

Finally we derive an expression for P.loss as a function of frequency ω.
Substituting C from (2) in the expression for P.loss (5) we obtain:

$$P.loss = E.stored*\left(\frac{L*\omega^2}{Rp}+\frac{Rs}{L}\right)$$

Substituting C from (2) in capacitor energy expression (3) we obtain:

$$E.stored = E.c = \frac{V^2}{2*L*\omega^2}$$

We can now write the expression for P.loss(ω) as:

$$P.loss(\omega)=\frac{V^2}{2*L*\omega^2}*\left(\frac{L*\omega^2}{Rp}+\frac{Rs}{L}\right)=\frac{V^2}{2*Rp}+\frac{V^2*Rs}{2*L^2}*\frac{1}{\omega^2}$$ (10), which has all constant terms except ω. Note that the first additive term is related to losses in Rp, the second additive term is related to losses in Rs and the terms are separable.

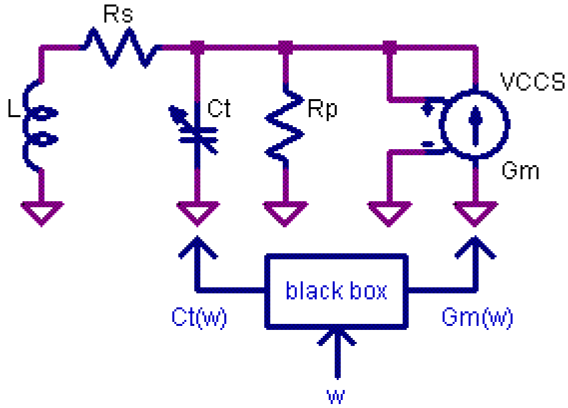
2. Properties of the capacitively tuned LC oscillators with Rs-dominant, Rp-dominant and mixed loss models under varying resonance frequency.

There is a simple analytical approximation allowing us to study characteristic dependence of oscillation condition on frequency that is reasonably accurate if: (i) unloaded Q of the resonant circuit including Rp-dominant or Rs-dominant loss model is sufficiently high, and (ii) we assume an ideal amplifier (a voltage controlled current source with variable Gm).

The model is based on the assumption that if $Q>>1$ then in a LC circuit comprised of a single inductor (the inductor may have taps/sections but no leakage between sections) and multiple capacitors (various capacitive dividers) the voltage at all circuit nodes will have have either approximately the same phase or phase shifted by 180 degree. Phase shifts and amplitude changes produced by the loss model resistors Rp/Rs and by the current injection from the amplifier will be very small compared to the recirculating currents in the LC network, due to high Q assumption. This allows using simple RMS calculations for the energy transformations in the circuit without resorting to vector operations and complex impedance calculus.

Class 1 - the "simple" oscillator:

The simplest oscillator model is illustrated on the diagram below. We have a capacitively tuned parallel resonant circuit LCt, either Rp or Rs loss model (both are shown on the diagram but we will analyze the cases separately), a non-inverting voltage controlled current source (VCCS) , and a black box A controlling Gm of the amplifier to maintain oscillation condition. The VCCS is sensing the resonant tank AC voltage and returning a small fraction of energy equal to the tank losses back to the tank.



We design the black box A to maintain constant energy in the resonance tank LCt instead of the usual constant amplitude condition. Then we change the oscillator frequency ω very slowly (varying Ct) and observe control signal generated by the black box to maintain constant energy in the tank as we tune. The experiment conditions can be summarized as follows:

1. *Q>>1*, therefore voltage phase at all nodes is either approximately the same phase or inverted;
2. *E.stored = Const;*
3. equation (1) is always satisfied as we vary *ω*, since we study the oscillator only at the resonance.
4. the variation of *ω* is slow

This way we will derive Gm dependence on a slowly varying parameter ω corresponding to a steady state condition for different oscillator classes.

Steady state oscillation requires that:

$$P.added = P.loss \quad (11)$$

We can calculate *P.added* by the amplifier as follows. The VCCS observes voltage *Vin* and generates the current at the output:

$Iout = Vin * Gm(\omega)$,where Gm(ω) is a transconductance gain of an ideal VCCS corresponding to the steady state oscillations at frequency ω.

Also note that for this simple oscillator:

$$Vin = Vout = V$$

Then, using the negligible phase shift assumption, we can easily estimate RMS power returned back to the tuned circuit (remember we are using peak V and I):

$$P.added = \frac{1}{2} Vout * Iout = \frac{1}{2} Vout * (Gm(\omega) * Vin) = \frac{1}{2} * V^2 * Gm(\omega) \quad (12)$$

Now for a steady state oscillation condition, combining (10), (11), (12) we can write:

$$\frac{1}{2} * V^2 * Gm(\omega) = \frac{V^2}{2 * Rp} + \frac{V^2 * Rs}{2 * L^2} * \frac{1}{\omega^2}$$

Simplifying the last expression <u>for the Rs+Rp mixed loss model we get:</u>

$$Gm(\omega)=\frac{1}{Rp}+\frac{Rs}{L^2}*\frac{1}{\omega^2} \quad (13)$$

The formula for Gm(ω) covers the general case of a mixed Rs+Rp loss model. We will now apply this formula to two boundary cases:

<u>For the Rp-dominant model we get:</u>

$$Gm(\omega)=\frac{1}{Rp}$$

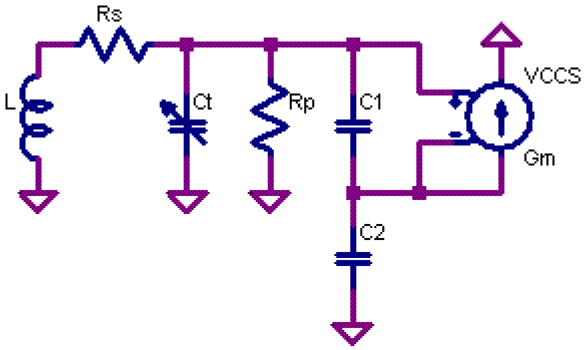
We can see that with Rp-dominant load model in a simple oscillator the gain Gm is not dependent on frequency ω, and the black box supplies fixed gain control voltage over the tuning span. In fact this simple oscillator model is representative for all class 1 oscillators. When presented with the Rp-dominant loss model these oscillators require little to no gain adjustment across the wide tuning range.

<u>Next, for the Rs-dominant model we get:</u>

$$Gm(\omega)=\frac{Rs}{L^2}*\frac{1}{\omega^2}$$

Here we can see that with the Rs-dominant load model the gain Gm(ω) is inversely proportional to the frequency squared, hence substantial downward adjustment of the gain is needed when the circuit is tuned towards the upper part of the tuning range. Again, this simple oscillator model is representative for all class 1 oscillators. When presented with the Rs-dominant loss model these oscillators require substantial reduction of gain as you tune up the band.

Class 1 - Colpitts oscillator:
We can easily augment the analysis of the simple oscillator above with the Colpitts capacitive divider if the divider capacitance values are sufficiently large and the resonant circuit Q is high enough such that recirculating currents in the divider itself are much larger than the amplifier injected current into the divider. This is critical for our main phase relationship assumption to hold. Only then the energy based calculations demostrated above can be directly applied to the Colpitts oscillator threshold analysis. Note that in a Colpitts oscillator the tank capacitor is comprised of a sum of Ct and the aggregate value of the serially connected C1 and C2, but this has no impact on our analysis.



The general formula for voltages developed across C1 and C2 in a capacitive divider C1/C2 are:

$Vc1 = Vin = V * \frac{C2}{C1+C2}$, $Vc2 = Vout = V * \frac{C1}{C1+C2}$, where V is the peak voltage developed across the inductor L, and Vin and Vout are the voltages seen by the amplifier input and output correspondingly.

Using the negligible phase shift assumption, we can estimate RMS power returned back to the tuned circuit via C2 in a Colpitts oscillator like we did in (12):

$$P.added = \frac{1}{2} * Vout * Iout = \frac{1}{2} * V * \frac{C1}{C1+C2} * \left(V * \frac{C2}{C1+C2} * Gm(\omega)\right) = \frac{1}{2} * \frac{C1*C2}{(C1+C2)^2} * V^2 * Gm(\omega) ,$$

where *Iout* is the current injected by the amplifier back to C2. We can see that the relationship is similar to (12) except for the constant multiplier $\frac{1}{A} = \frac{C1*C2}{(C1+C2)^2}$, or $A = \frac{(C1+C2)^2}{C1*C2}$ (14)

Now for a steady state oscillation condition (11) we can write:

$$\frac{1}{2} * \frac{1}{A} * V^2 * Gm(\omega) = \frac{V^2}{2*Rp} + \frac{V^2 * Rs}{2*L^2} * \frac{1}{\omega^2}$$

Simplifying we arrive at a general form for Gm(ω) that will be used for all oscillators in this study:

$$Gm(\omega) = A * \left(\frac{1}{Rp} + \frac{Rs}{L^2} * \frac{1}{\omega^2}\right) \quad (15)$$

For a Colpitts oscillator using the term A from (14) for the Rs+Rp mixed loss model we get:

$$Gm(\omega) = \frac{(C1+C2)^2}{C1*C2} * \left(\frac{1}{Rp} + \frac{Rs}{L^2} * \frac{1}{\omega^2}\right)$$

For the Rp-dominant model we get:

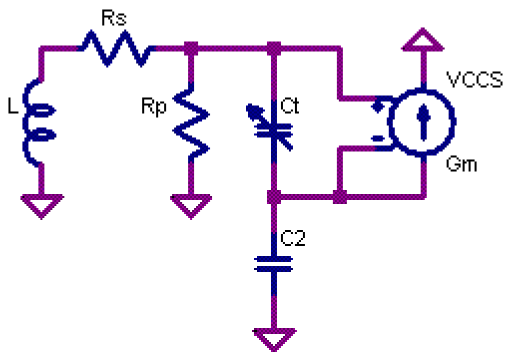
$$Gm(\omega) = \frac{(C1+C2)^2}{C1*C2} * \frac{1}{Rp}$$

For the Rs-dominant model we get:

$$Gm(\omega)=\frac{(C1+C2)^2}{C1*C2}*\frac{Rs}{L^2}*\frac{1}{\omega^2}$$

From the last formulas we can see that the Colpitts oscillator Gm dependence on the frequency ω is exactly the same (up to a constant term) as for our simple oscillator model above.

Class 2 - simplified Vackar oscillator:
For this analysis we use the simplified Vackar-like oscillator shown below. Note that the usual Vackar fixed capacitive divider at the amplifier input is omitted and the circuit is re-drawn to resemble a Colpitts oscillator, except that the divider now includes tuning capacitor Ct instead of C1.



In a Vackar oscillator C2>>Ct, where the the capacitive divider multiplicative term A (14) can be approximated as:

$$A = \frac{(Ct+C2)^2}{Ct * C2} \approx \frac{C2}{Ct}$$

As opposed to the fixed capacitive divider in the Colpitts oscillator, this term is frequency dependent and we are interested in how it changes with ω. Generally we must use the value of the series connected Ct and C2 $\frac{Ct * C2}{Ct+C2}$ in the resonance condition (1),(2). However since C2>>Ct we can ignore the effect of C2, so we just substitute Ct from (2) into the expression for the term A and obtain:

$$A \approx \frac{C2}{Ct} \approx C2 * L * \omega^2 \quad (16)$$

We can now re-write (15) using term A from the above.

For the Rs+Rp mixed loss model we get:

$$Gm(\omega) \approx C2 * L * \omega^2 * \left(\frac{1}{Rp} + \frac{Rs}{L^2} * \frac{1}{\omega^2}\right)$$

For the Rp-dominant model we get:

$$Gm(\omega) \approx \frac{C2 * L}{Rp} * \omega^2$$

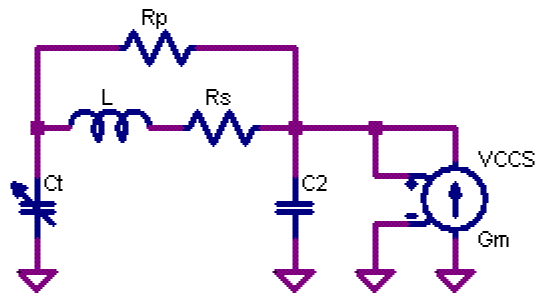
We can conclude that for the Rp-dominant model in the simplified Vackar oscillator the amplifier Gm must grow approximately proportional to the frequency squared.

For the Rs-dominant model we get:

$$Gm(\omega) \approx \frac{C2 * Rs}{L}$$

We can conclude that for the Rs-dominant model in the simplified Vackar oscillator Gm stays nearly constant with the frequency change. This agrees completely with the simulation results. When presented with the Rp-dominant loss model all class 2 oscillators require little to no gain adjustment across the wide tuning range.

Class 3 - simplified Clapp-Gouriet oscillator:
For the circuit analysis of the C-G oscillator we use the simplified model, the same way we did for the first class 1 oscillator. The divider at the amplifier input is omitted, therefore the model will be accurate up to a constant, frequency invariant term. The VCCS is sensing the voltage across the capacitor C2 and returning a small fraction of energy equal to the tank losses back to the tank via the same path. The next diagram illustrates the model circuit:



The capacitors Ct and C2 form a divider. Applying the capacitive divider formula like we did in the Colpitts oscillator analysis we can write:

$Vin=Vout=V*\frac{Ct}{Ct+C2}$, where V is the peak voltage developed across the inductor L, and Vin and Vout are the voltages seen by the amplifier input and output correspondingly.
Therefore the expression for P.added can be written as:

$$P.added=\frac{1}{2}*Vout*Iout=\frac{1}{2}*V*\frac{Ct}{Ct+C2}*(V*\frac{Ct}{Ct+C2}*Gm(\omega))=\frac{1}{2}*\frac{Ct^2}{(Ct+C2)^2}*V^2*Gm(\omega)$$

Therefore the term A in the general expression (15) can be written as:

$A=\frac{(Ct+C2)^2}{Ct^2}$, since C2>>Ct we can approximate $(Ct+C2)^2$ with $C2^2$ and write $A\approx\frac{C2^2}{Ct^2}$

Again the term is frequency dependent. We are interested in how this term changes with ω, so ignoring the effect of C2 on the tuning frequency as before we substitute Ct from (2) corresponding to the resonance condition and get:

$$A\approx C2^2*L^2*\omega^4$$

We can now re-write (15) using the term A from the expression above.
For the Rs+Rp mixed loss model we get:

$$Gm(\omega)\approx C2*L^2*\omega^4*(\frac{1}{Rp}+\frac{Rs}{L^2}*\frac{1}{\omega^2})$$

For the Rp-dominant model we get:

$$Gm(\omega)\approx\frac{C2*L^2}{Rp}*\omega^4$$

We can see that for the Rp-dominant model in the simplified C-G oscillator the amplifier Gm must grow approximately proportional to the 4th power of the frequency.
For the Rs-dominant model we get:

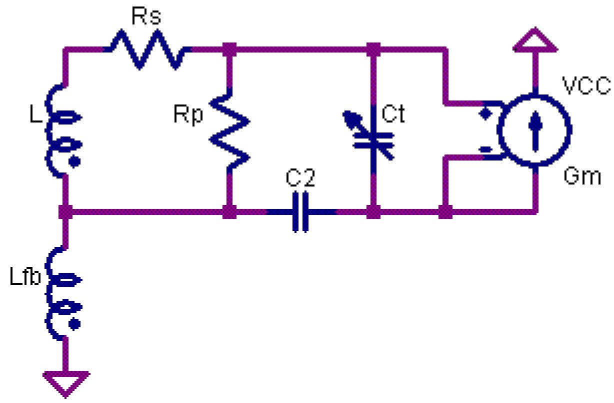
$$Gm(\omega)\approx C2*Rs*\omega^2$$

We can see that for the Rp-dominant model in the simplified C-G oscillator the amplifier Gm must grow approximately proportional to the frequency squared.

Both mathematical models agree with the simulation results.

Hybrid feedback class - Hartley-Vackar oscillator:

For a hybrid feedback analysis we use a simplified Hartley-Vackar oscillator because it allows for better re-use of the previous calculations. The circuit diagram is shown below:



This oscillator constructively combines two feedback paths. The current produced by the amplifier is injected into C2 like in the simplified Vackar oscillator described above, but the very same current is also injected into the feedback coil Lfb. It is important to note that the resonant circuit does not include Lfb, all recirculating currents are still contained in L/Ct/C2 loop. This means that (i) Lfb has no effect on the resonance frequency, (ii) the voltage developed across Lfb is not used to calculate *E.stored*, and (iii) Rp loss model resistor is connected across the resonant section only. Assuming no flux leakage between inductor segments, the voltage developed across Lfb is directly proportional to the voltage developed across the inductor L and has exactly the same phase.

The voltage seen by the amplifier across Ct can be estimated exactly as in the Colpitts oscillator analysis, while the voltage excursion at the amplifier output is a sum of the voltages developed across C2 and Lfb:

$Vin = V*\frac{C2}{Ct+C2}$, and $Vout = V*\frac{Ct}{Ct+C2}+V*K = V*(\frac{Ct}{Ct+C2}+K)$, were K is the turn ratio of *Lfb* to *L*. Using the negligible phase shift assumption we can estimate the RMS power returned back to the tuned circuit via C2 and Lfb in this hybrid oscillator like we did in (12):

$$P.added = \frac{1}{2}*Vout*Iout = \frac{1}{2}*V*(\frac{Ct}{Ct+C2}+K)*(V*\frac{C2}{Ct+C2}*Gm(\omega))$$

$$P.added = \frac{1}{2}*\frac{Ct*C2*(1+K)+C2^2*K}{(Ct+C2)^2}*V^2*Gm(\omega)$$

Since $K<<1$ we can replace $1+K$ with 1, and also since in the Vackar feedback $C2>>Ct$ we can replace $(Ct+C2)^2$ with $C2^2$. Simplifying we can write:

$$P.added \approx \frac{1}{2}*\frac{Ct+C2*K}{C2}*V^2*Gm(\omega)$$

Note that we can not discard term $C2*K$ in the approximation above despite $K<<1$ because $C2>>Ct$. We can write the multiplicative term A as:

$$\frac{1}{A} \approx \frac{Ct+C2*K}{C2} \qquad A \approx \frac{C2}{Ct+K*C2}$$

Again the term A is frequency dependent. We are interested in how this term changes with *ω,* so ignoring the effect of C2 on the tuning frequency as we did before we substitute Ct from (2), corresponding to the resonance condition and get:

$$A \approx \frac{C2}{Ct + C2 * K} \approx \frac{C2}{\dfrac{1}{L * \omega^2} + C2 * K} = \frac{C2 * L * \omega^2}{1 + K * C2 * L * \omega^2} \quad (17)$$

We can now re-write (15) using the term A from the above.

<u>For the Rs+Rp mixed loss model we get:</u>

$$Gm(\omega) \approx \frac{C2 * L * \omega^2}{1 + K * C2 * L * \omega^2} * \left(\frac{1}{Rp} + \frac{Rs}{L^2} * \frac{1}{\omega^2}\right)$$

<u>For the Rp-dominant model we get:</u>

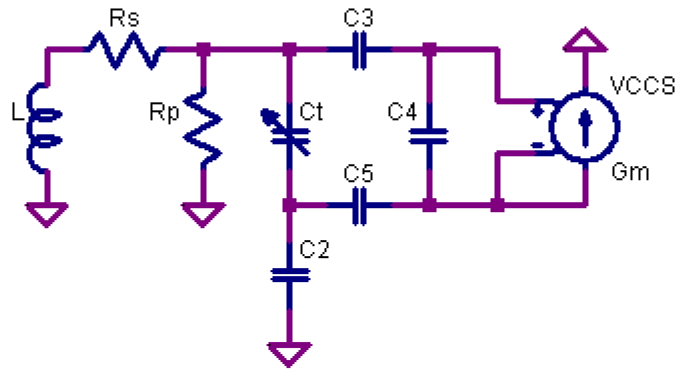
$$Gm(\omega) \approx \frac{C2 * L * \omega^2}{1 + K * C2 * L * \omega^2} * \frac{1}{Rp}$$

<u>For the Rs-dominant model we get:</u>

$$Gm(\omega) \approx \frac{C2}{1 + K * C2 * L * \omega^2} * \frac{Rs}{L}$$

The last two expressions are rational functions for variable ω and both functions are in the transition towards asymptotic region. For Rs-dominant loss model Gm(ω) is clearly decreasing with frequency but for Rp we need to explore it's graph to determine the behavior.

Hybrid feedback class - Seiler-Vackar oscillator:
The derivation of Gm(ω) for the Seiler-Vackar oscillator is slightly longer due to the extra three-capacitor divider but the method is the same and the end results are similar to the Hartley-Vackar above. This oscillator uses the capacitive-only feedback in both paths and should be less prone to parasitic oscillations at higher frequencies, therefore it deserves a separate analysis. The circuit diagram for the Seiler Vackar oscillator is shown below.



This oscillator constructively combines two feedback paths. The VCCS amplifier senses the voltage developed across C4. The current produced by the amplifier is injected into C2 like in the simplified Vackar oscillator, but the very same current is also injected into the capacitive divider formed by C3/C4/C5 like in the conventional Seiler oscillator.

The capacitive divider C3/C4/C5 is connected in parallel to Ct and affects the tuning frequency, hence we introduce the equivalent capacitance Cs equal to the sum of Ct and the series connected divider C3/C4/C5. The equivalent capacitance Cs is connected in series with C2. The voltages at both dividers Cs/C2 and C3/C4/C5 are frequency dependent and interrelated so we can not immediately discard the extra capacitance of the divider like we did in the Colpitts oscillator analysis before.

Using the capacitive divider rule for Cs/C2 divider we can write:

$$V_{Cs}=\frac{C2}{Cs+C2}*V \quad , \quad V_{C2}=\frac{Cs}{Cs+C2}*V \quad \text{, where} \quad Cs=\frac{C3*C4*C5}{C3*C4+C3*C5+C4*C5}+Ct$$

For C3/C4/C5 divider we can obtain:

$$Vin=V_{C4}=K1*V_{Cs}=K1*\frac{C2}{Cs+C2}*V \quad \text{, where} \quad K1=\frac{C3*C5}{C3*C4+C3*C5+C4*C5}$$ is a fraction of the voltage across Ct that appears across C4.

$$V_{C5}=K2*V_{Cs}=K2*\frac{C2}{Cs+C2}*V \quad \text{,where} \quad K2=\frac{C3*C4}{C3*C4+C3*C5+C4*C5}$$ is a fraction of the voltage across Ct that appears across C5.

The total Vout voltage is a sum of two voltages:

$$Vout=V_{C2}+V_{C5}=\frac{Cs}{Cs+C2}*V+K2*\frac{C2}{Cs+C2}*V=(\frac{Cs+K2*C2}{Cs+C2})*V$$

Now the expression for P.added can be written as:

$$P.added=\frac{1}{2}*Vout*Iout=\frac{1}{2}*\frac{Cs+K2*C2}{Cs+C2}*V*(\frac{K1*C2}{Cs+C2}*V*Gm(\omega)) \quad \text{, simplifying we get}$$

$$P.added=\frac{1}{2}*\frac{(Cs+K2*C2)*K1*C2}{(Cs+C2)^2}*V^2*Gm(\omega)$$

Therefore the term A in the general expression (15) can be written:

$$A=\frac{(Cs+C2)^2}{K1*C2*(Cs+K2*C2)}$$

Since C2>>Cs we can approximate $(Ct+C2)^2$ with $C2^2$ and write:

$$A\approx\frac{C2}{K1*(Cs+K2*C2)}$$

Again this term is frequency dependent. We are interested in how this term changes with ω, so ignoring the effect of C2 on the tuning frequency as we did before we substitute Cs from (2) corresponding to the resonance condition and get an approximation:

$$A\approx\frac{C2}{K1*(\frac{1}{L*\omega^2}+K2*C2)}=\frac{C2*L*\omega^2}{K1*(1+K2*C2*L*\omega^2)}$$

We can now re-write (15) using the term A from the above.

For the Rs+Rp mixed loss model we get:

$$Gm(\omega)\approx\frac{C2*L*\omega^2}{K1*(1+K2*C2*L*\omega^2)}*(\frac{1}{Rp}+\frac{Rs}{L^2}*\frac{1}{\omega^2})$$

For the Rp-dominant model we get:

$$Gm(\omega)\approx\frac{C2*L*\omega^2}{K1*(1+K2*C2*L*\omega^2)}*\frac{1}{Rp}$$

For the Rs-dominant model we get:

$$Gm(\omega)\approx\frac{C2}{K1*(1+K2*C2*L*\omega^2)}*\frac{Rs}{L}$$

We can see that these expressions closely resemble the Hartley-Vackar oscillator results. The differences being: (i) the extra constant factor $\frac{1}{K1}$, and (ii) the definition of K2.

Gm(ω) exhibits exactly the same behavior in the frequency domain (up to a constant factor), hence for details we can simply refer to the analytical graphs done for the Hartley-Vackar circuit.

LC network reversibility and duals.

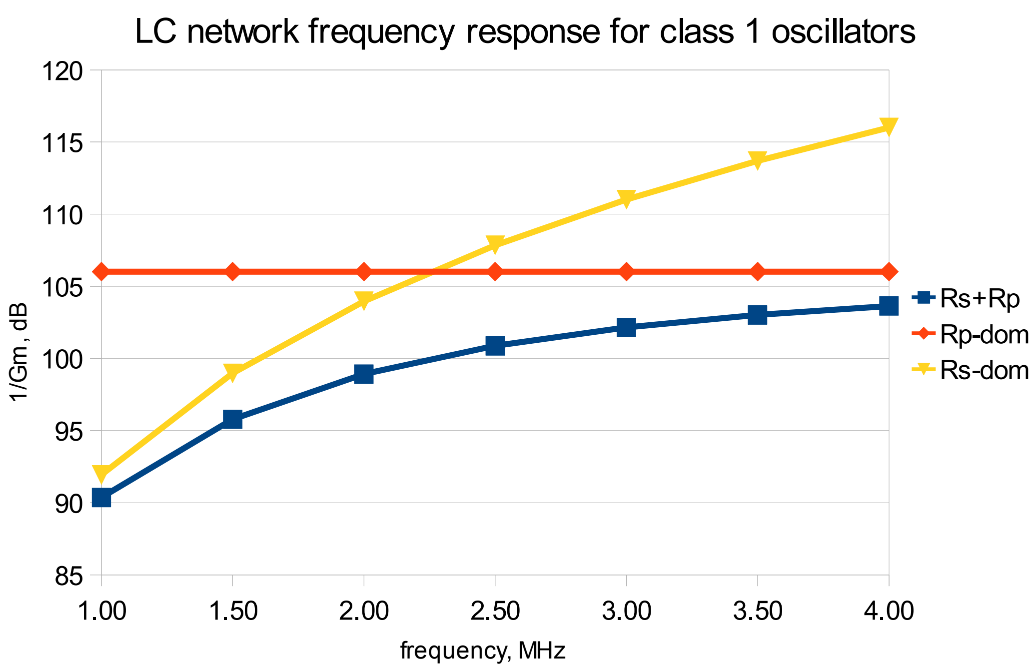
All LC networks in this paper are reversible - the frequency domain response and the threshold behavior is not changed if the network input and output are swapped. In the simplified variants of the class 1 and class 3 oscillators the reversibility is obvious since $Vin = Vout$. In the simplified Vackar oscillator (class 2) it is possible to swap C2 and Ct, yet the oscillator threshold behavior will not change. To illustrate this less obvious fact consider that Vin, Vout are simple factors of P.added for all oscillators described in this paper. If the expressions for Vout and Vin are swapped in the formula for P.added - the result does not change, and Gm(ω) formulas stay unchanged as well.

The reversibility means that with an ideal VCCS amplifier there are twice as many variations possible, each oscillator has a dual (although in some cases the dual pairs are identical). However when the LC network is matched with a real amplifier that has specific finite input and output impedance these duals will behave quite differently and selecting the right one requires engineering analysis. For example if a bipolar transistor (having a low active input impedance) is used in a simplified Vackar oscillator a dual offers a better amplifier-network impedance match.

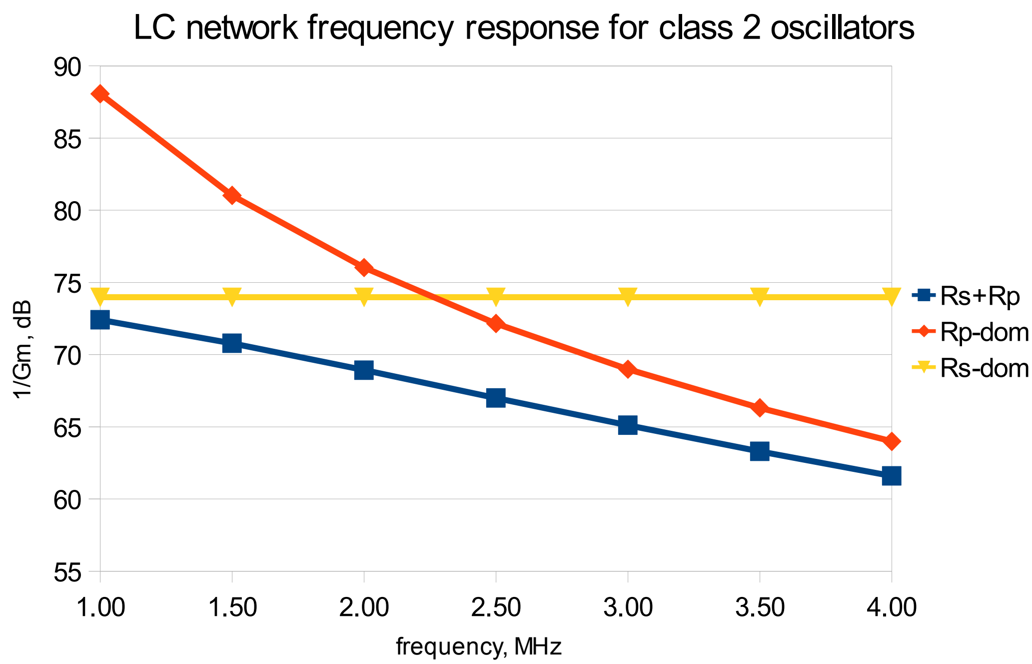
3. Verification of analytical formulas vs simulation results.

The analytical expressions for three oscillator types: class 1, class 2 and a hybrid class were entered into the spreadsheet to verify the theory against the simulations for three loss models: Rp-dominant, Rs-dominant and Rs+Rp mixed loss model. The resonant circuit parameters, loss parameters and test frequencies were chosen to match the simulation. Each graph below represents one oscillator class evaluated using three loss models. Note that the formulas produce steady state Gm(ω) which is the inverse of the LC network frequency response. To make the graphs comparable to the simulation an extra step was performed to calculate the quantity: $20*\log(\frac{1}{Gm(\omega)})$. The absolute value on the log scale (overall gain) was not matched to the simulation but on the relative scale the graphs are in perfect agreement with the simulation.

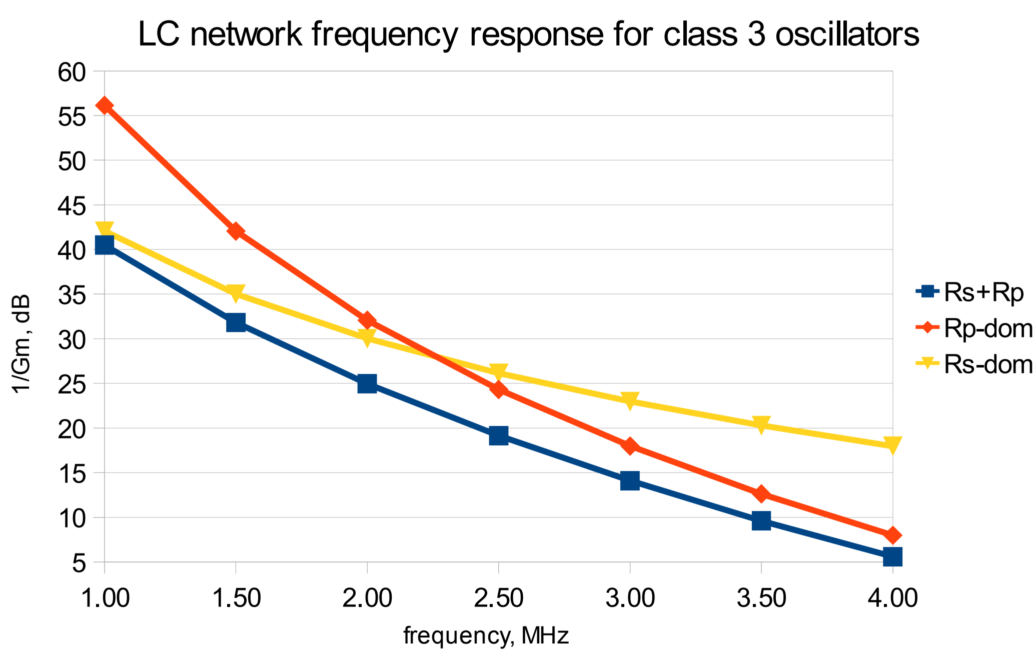
The first graph shows a simple class 1 oscillator analysis described in the beginning of this section and it should be compared to the class 1 simulation series. Model for model it matches the tilt direction, the magnitude of the change and even the convex character of non-linearity.



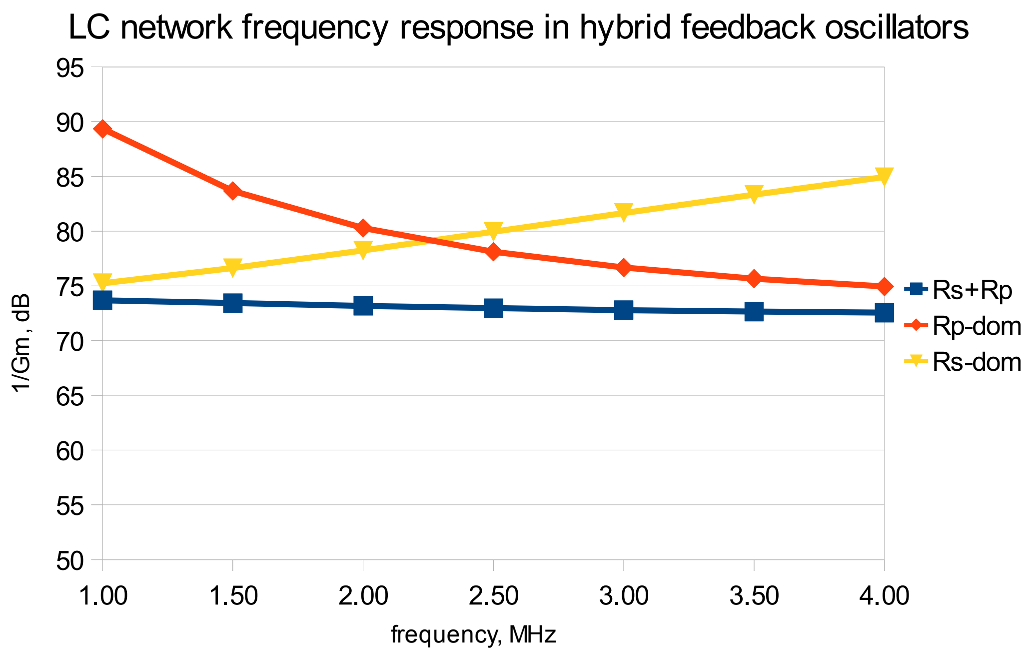
The second graph shows a simplified Vackar class 2 oscillator analysis and should be compared to the class 2 simulation series. Again the match is nearly perfect, reflecting concave character of non-linearity.



The next graph shows a simplified Clap-Gouriet oscillator and and should be compared to the class 3 simulation series. Similar to the simulation results this graph shows very strong frequency dependence at all three loss models.



Finally the last graph shows the simplified Hartley-Vackar oscillator analysis and should be compared to the hybrid class simulation series. Again we can a observe a nearly perfect match. Very slight amount of uncompensated tilt can be noticed in the mixed loss model but it can also be seen on the zoomed-out simulation graph in LTspice and is the result of the empirical fitting feedback parameters to the test Rs and Rp values. Analytical method for calculating the feedback parameters for optimal fit is described in the next section and results in even flatter response (within limitations of the simplified loss models and approximations used throughout the paper).



It should be noted that the empirical models provide a very good match to the simulation results despite fairly aggressive use of approximations in the derivation of some models.

4. Practical implications.

While this analysis explains nicely the simulation behavior and justifies the idea behind the hybrid feedback oscillators it is difficult to apply it in a practical oscillator design. The main problem is that in a real resonance circuit it is very hard to accurately estimate intrinsic loss parameters Rp and Rs and their dependence on frequency. Both intrinsic loss parameters Rs and Rp reflect actual physical phenomena happening in the resonant circuit but unfortunately these parameters are not easily observable.

There are two possible approaches for a practical design:
1. Guessing (based on experience) if the particular resonant circuit will be closer to the Rp-dominant or closer to the Rs-dominant loss model and balancing the feedback paths accordingly. For example the SW VFO/regen described in the [main] paper is clearly closer to the Rp-dominant model and requires small amount of class 2 feedback for tilt compensation. On the other hand the BCB regen is in-between those two models and the two feedback paths should be balanced.

2. Measuring Q(ω) of a resonant circuit directly at two or more frequency points and using Q(ω) function to design the optimum feedback. As opposed to intrinsic loss parameters Rs and Rp, we can observe and measure Q(ω), although this approach may not always be accessible since it requires Q-meter or network analyzer.

Let's rewrite energy balance equation using Q(ω) instead of Rs and Rp. From (6) we get:

$P.loss = \frac{\omega * E.stored}{Q(\omega)}$, then using (3) we can write:

$P.loss = \frac{\omega * C * V^2}{2 * Q(\omega)}$, we want to remove the dependence on Ct so we use (2) and get:

$P.loss = \frac{1}{2} * V^2 * \frac{1}{L * \omega * Q(\omega)}$, using energy balance equation (11) and the oscillator-specific multiplicative factor A we get:

$\frac{1}{2} * \frac{1}{A} * V^2 * Gm(\omega) = \frac{1}{2} * V^2 * \frac{1}{L * \omega * Q(\omega)}$, simplifying this equation we arrive at:

$$Gm(\omega) = A * \frac{1}{L * \omega * Q(\omega)}$$

In case of a Hartley-Vakcar oscillator:

$A \approx \frac{C2 * L * \omega^2}{1 + K * C2 * L * \omega^2}$,therefore

$$Gm(\omega) \approx \frac{C2 * \omega}{1 + K * C2 * L * \omega^2} * \frac{1}{Q(\omega)}$$

The problem now can be formulated like this: given measured Q(ω), L, and the amplifier target Gm_0, find such values of K and C2 that minimize the expression below over the given frequency range ω1<ω<ω2 using some reasonable criteria, for example least squares:

$$\min_{\omega 1 < \omega < \omega 2} \left(\frac{C2 * \omega}{1 + K * C2 * L * \omega^2} * \frac{1}{Q(\omega)} - Gm_0 \right)^2$$

This problem above can be solved numerically on a computer, taking advantage of more than two measured points of $Q(\omega)$ in the optimization.

However we can consider a simplified strategy using just two measurements of $Q(\omega)$ at two frequencies ω_0 and ω_1, located close to the upper and the lower ends of the target tuning range. Let's say the measured Q-factors are: q_0 at ω_0, and q_1 at ω_1.
We want to satisfy two conditions:
1. Equal gain at the two frequencies:
$$Gm(\omega_o)=Gm(\omega_1)$$
2. The gain should be equal to the amplifier target gain Gm_0:
$$Gm(\omega_o)=Gm_0$$
This gives us a system of two equations to solve.
Solving the first equation (expressing K in terms of C2):
$$\frac{C2*\omega_0}{1+K*C2*L*\omega_0^2}*\frac{1}{q_0}=\frac{C2*\omega_1}{1+K*C2*L*\omega_1^2}*\frac{1}{q_1}$$
and skipping intermediate steps we get:
$$K=t*\frac{1}{L}*\frac{1}{C2} \quad \text{,where} \quad t=\frac{q_0*\omega_1-q_1*\omega_0}{q_1*\omega_0*\omega_1^2-q_0*\omega_1*\omega_0^2} \quad (20)$$
Now we substitute K into the second equation:
$$\frac{C2*\omega_0}{1+t*\omega_0^2}*\frac{1}{q_0}=Gm_0 \quad \text{,we can find the value C2 as:}$$
$$C2=Gm_0*q_0*\frac{1+t*\omega_0^2}{\omega_0} \quad (21)$$
Finally we can express K in terms of our design parameters:
$$K=\frac{1}{L*Gm_0*q_0}*\frac{t*\omega_0}{1+t*\omega_0^2} \quad (22)$$

For verification purposes the formulas (20),(21) and (22) were entered into the spreadsheet. The same design parameters were used as in the simulation runs as well as for generating graphs in the previous subsection:
1. f0=1MHz, f1=4MHz
2. L=100uH
3. q0=Q(f0) was calculated using formula (8), using Rs=10, Rp=2e5
4. q1=Q(f1) was calculated using formula (8), using Rs=10, Rp=2e5
5. amplifier target Gm0=0.2mS
Note that Gm0 target is rather low for two reasons - (i) the calculation is done for the simplified version with no Vackar-style capacitive divider at the amplifier input, and (ii) resonance tank Q is fairly high (>50) with parameters used for simulation.

The spreadsheet solver generated the optimum tilt compensation values for the Hartley-Vackar oscillator: C2=2nF, K=0.025, which yielded even better tilt compensation than the pair C2=2nF, K=0.02 empirically chosen during the simulation. Varying the target Gm0 at the solver input produced different C2/K value pairs exhibiting zero tilt in the simulation.